AXIALLY SYMMETRIC STEFAN PROBLEM WITH A BOUNDARY CONDITION OF THE THIRD KIND

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The solution of an axially symmetric single-phase Stefan problem with a boundary condition of the third kind is reduced to the solution of a nonlinear integrodifferential equation, the latter solution being obtained with the aid of the method of successive approximations.

An effective method for obtaining solutions of two-dimensionally, axially, and spherically symmetric single-phase one-dimensional Stefan problems concerning freezing, with conditions on the known boundary being constant with time and with zero initial expansion of the domain in which the solution is sought, is found to be one in which the Stefan problem is reduced to a nonlinear integrodifferential equation with a subsequent application of an iterational process [1-4].

We consider the following problem:

$$\rho c \frac{\partial T}{\partial \tau} = k \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) \quad \text{in} \quad D^*, \tag{1}$$

$$k \frac{\partial T}{\partial r} = h_w(T - T_0) \quad \text{for} \quad r = a, \tag{2}$$

$$k \frac{\partial T}{\partial r} = h_l(T_l - T_f) + \rho L \frac{d\delta}{d\tau} \quad \text{for} \quad r = \delta(\tau), \tag{3}$$

$$T = T_f = \text{const} \quad \text{for} \quad r = \delta(\tau), \tag{4}$$

$$\delta = a \quad \text{for} \quad \tau = 0. \tag{5}$$

Here $D^* = \{r, \tau: a < r < \delta(\tau),\ 0 < \tau < M < \infty\}$ defines the domain of variation of the variables in which the solution of the problem is to be determined; a is the radius of a cylindrical tube along which a cold liquid with temperature T_0 = const flows; h_w is the heat transfer coefficient of the cold liquid with the bounding surface; T_l, a constant, is the temperature of the liquid washing the tube ($T_l \geq T_f$); $r = \delta(\tau)$ is the time-dependent position of the phase-change boundary. The boundary condition in the form (3) makes it possible to take into account the influence of the convective heat transfer on the intensity of growth of the "solid" phase.

We introduce the dimensionless quantities:

$$x = \frac{r}{a}; \quad t = \frac{k}{\rho c a^2}\tau; \quad \Delta = \frac{\delta}{a}; \quad u = \frac{T - T_f}{T_0 - T_f}.$$

We rewrite the problem (1)-(5) in dimensionless form:

$$\frac{\partial u}{\partial t} = \frac{1}{x} \cdot \frac{\partial}{\partial x} x \frac{\partial u}{\partial x} \quad \text{in} \quad D, \tag{6}$$

$$\frac{\partial u}{\partial x} = \alpha(u - 1) \quad \text{for} \quad x = 1, \tag{7}$$

$$\frac{\partial u}{\partial x} = \beta + \varphi \frac{d\Delta}{dt} \quad \text{for} \quad x = \Delta(t), \tag{8}$$

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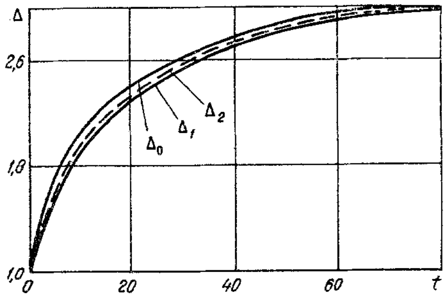

Fig. 1

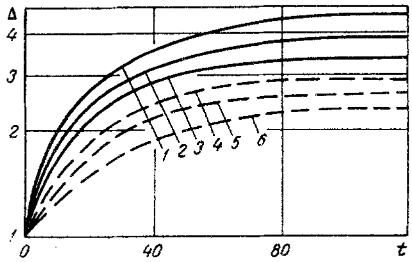

Fig. 2

Fig. 1. Nature of the convergence of the iterational process. The subscripts 0, 1, 2 refer, respectively, to the results of the zeroth, the first, and the second approximation ($\alpha = 2$; $\beta = 0.5$; $\varphi = -5$).

Fig. 2. Position of the phase transition boundary versus the time (second approximation results): Curves 1 and 4, $\beta = 0.05$; Curves 2 and 5, $\beta = -0.10$; Curves 3 and 6, $\beta = -0.15$; Dashed line, $\alpha = 0.5$; Solid line, $\alpha = 2.0$.

$$u = 0 \quad \text{for} \quad x = \Delta(t), \tag{9}$$

$$\Delta = 1 \quad \text{for} \quad t = 0, \tag{10}$$

where D is the dimensionless analog of the domain D*; $\alpha = ah_w/k$ is a dimensionless parameter of the problem characterizing the heat transfer intensity on the tube surface; $\beta = ah_l/k \cdot T_l - T_f/T_0 - T_f$ is a dimensionless parameter of the problem characterizing the convective heat transfer intensity on the phase-change surface; $\varphi = L/c(T_0 - T_f)$ is the dimensionless heat of phase transition.

We note that even when $\beta = 0$ the problem (6)-(10) has no self-similar solution.

We now construct the solution of problem (6)-(10). Integrating Eq. (6) twice with respect to the variable x between the limits of 1 and x and using the conditions (7) and (9), we obtain, after making simplifications,

$$u = -\alpha \ln x + \frac{1 + \alpha \ln x}{1 + \alpha \ln \Delta}\left[\alpha \ln \Delta - \int_1^\Delta \frac{1}{x}\int_1^x x \frac{\partial u}{\partial t}\, dx dx\right] + \int_1^x \frac{1}{x}\int_1^x x \frac{\partial u}{\partial t}\, dx dx. \tag{11}$$

Invoking the condition (8), we write

$$-\frac{\alpha}{\Delta}\left[1 - \frac{\alpha \ln \Delta - \int_1^\Delta \frac{1}{x}\int_1^x x \frac{\partial u}{\partial t}\, dx dx}{1 + \alpha \ln \Delta}\right] + \frac{1}{\Delta}\int_1^\Delta x \frac{\partial u}{\partial t}\, dx = \beta + \varphi \frac{d\Delta}{dt}.$$

Noting that $\partial/\partial t = \partial/\partial\Delta \cdot d\Delta/dt$, we solve the resulting relationship for $\dot{\Delta} = d\Delta/dt$:

$$\dot{\Delta} = \frac{\alpha + \beta\Delta(1 + \alpha \ln \Delta)}{(1 + \alpha \ln \Delta)\int_1^\Delta x \frac{\partial u}{\partial \Delta}\, dx - \alpha \int_1^\Delta \frac{1}{x}\int_1^x x \frac{\partial u}{\partial \Delta}\, dx dx - \varphi\Delta(1 + \alpha \ln \Delta)}. \tag{12}$$

The limiting thickness of the frozen layer Δ_s is determined from the condition $\dot{\Delta} = 0$ and can be calculated as the root of the transcendental equation

$$\Delta_s(1 + \alpha \ln \Delta_s) = -\frac{\alpha}{\beta}. \tag{13}$$

Taking the relation (12) into account, we can rewrite the expression (11) in the form

$$u = Au, \tag{14}$$

where the operator A acts on the function $u(x, \Delta)$ according to the rule

$$Au = \alpha \ln \Delta \frac{1 + \alpha \ln x}{1 + \alpha \ln \Delta} - \alpha \ln x$$

$$+ \frac{[\alpha + \beta\Delta(1 + \alpha \ln \Delta)] \left[\frac{1 + \alpha \ln x}{1 + \alpha \ln \Delta} \int\limits_1^{\Delta} \frac{1}{x} \int\limits_1^{x} x \frac{\partial u}{\partial \Delta} dx dx - \int\limits_1^{x} \frac{1}{x} \int\limits_1^{x} x \frac{\partial u}{\partial \Delta} dx dx \right]}{\varphi\Delta(1 + \alpha \ln \Delta) + \alpha \int\limits_1^{\Delta} \frac{1}{x} \int\limits_1^{x} x \frac{\partial u}{\partial \Delta} dx dx - (1 + \alpha \ln \Delta) \int\limits_1^{\Delta} x \frac{\partial u}{\partial \Delta} dx}.$$

We construct the solution of Eq. (14) in accord with the iterational scheme

$$u_{k+1} = Au_k, \quad k \geqslant 0, \tag{15}$$

taking as the zeroth approximation the expression

$$u_0 = \alpha \left[\frac{1 + \alpha \ln x}{1 + \alpha \ln \Delta} \ln \Delta - \ln x \right]. \tag{16}$$

Noting that the right side of Eq. (12) does not contain the time explicitly and taking into account the initial condition (10), we obtain

$$t = Bu, \tag{17}$$

where the operator B is defined as follows:

$$Bu = \int\limits_1^{\Delta} \frac{(1 + \alpha \ln \Delta) \int\limits_1^{\Delta} x \frac{\partial u}{\partial \Delta} dx - \alpha \int\limits_1^{\Delta} \frac{1}{x} \int\limits_1^{x} x \frac{\partial u}{\partial \Delta} dx dx - \varphi\Delta(1 + \alpha \ln \Delta)}{\alpha + \beta\Delta(1 + \alpha \ln \Delta)} d\Delta. \tag{18}$$

We define the time taken for the desired phase transition boundary to reach a given position, for the corresponding approximations (15), by the relation

$$t_k = Bu_k.$$

The expressions (17), and, correspondingly, the expressions (18), determine, in general, a function $t = t(\Delta)$, inverse to the desired function; however, in physically real cases, there is a one-to-one correspondence between the direct function $\Delta = \Delta(t)$ and its inverse.

The convergence of the iterational process according to the scheme (15)-(18) in the range of variation considered for the dimensionless parameters of the problem is found to be completely satisfactory; this is evident from Fig. 1 where the calculated curves $\Delta = \Delta(t)$ are shown for the zeroth, first, and second approximations for one of the sets of values of α, β, φ. The relative difference in the results of the second and first approximations amounts to a value of 1.5-2%.

Figure 2 shows the results obtained for $\Delta = \Delta(t)$ using the second approximation for several values of the defining parameters of the problem.

NOTATION

ρ is the density of medium;
c is the specific heat capacity of medium;
T is the temperature;
τ is the time;
k is the thermal conductivity;
r is the space coordinate;
D is the range of variable variation;
h_w is the heat transfer coefficient from tube wall to coolant;
T_0 is coolant temperature;
h_l is the heat transfer coefficient between liquid phase and phase transition surface;
T_f is the temperature at the boundary of phase transition;
L is the heat of phase transition;
δ is the thickness of frozen solid layer;
x is the dimensionless space coordinate;

Δ is the dimensionless thickness of the frozen ice layer;
u is the dimensionless temperature;
α is the heat transfer rate on the tube surface;
B is the convective heat transfer rate at phase transition interface;
φ is the dimensionless heat of phase transition;
Δ_S is the limit relative value of frozen layer;
A is the integro-differential operator;
B is the operator.

Subscripts

k = 0, 1, 2 means the result of appropriate approximation.

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